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Existence of zero points for pseudomonotone operators in Banach spaces

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Abstract The purpose of this paper is to study the existence of zero points for set-valued pseudomonotone operators in a Banach space by using a new condition which was recently proposed by the authors (Matsushita and Takahashi, Set-Valued Analysis 15:251–264, 2007).

Keywords Normal cone · Zero point · Pseudomonotone · Upper hemicontinuous · Variational inequality

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1 Introduction

Let C be a closed convex subset of a Banach space E, let E^* be the dual of E and let $T : C \to 2^{E^*}$ be a set-valued operator. Then an element $x_0 \in C$ is said to be a zero point of T if $0 \in Tx_0$. The existence of zero points for set-valued operators is closely related to the following variational inequality problem: Find $u \in C$ and $u^* \in Tu$ such that

$$\langle y - u, u^* \rangle \geq 0 \quad \text{for all} \quad y \in C. \tag{1.1}$$

The variational inequality problem is important in many applications. We refer to Kinderlehrer and Stampacchia [17], Aubin [4] and Facchinei and Pang [11] and references therein. Various conditions under monotonicity or generalized monotonicity assumption for T have been proposed by many researchers in order to guarantee solutions for variational inequality problems,

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see e.g. [7–10,13,15,16,22,23,26,27]. In most cases, T was assumed to be pseudomonotone (in the sense of Karamardian [16]).

On the other hand, the authors [20] (see also [14,19]) recently proposed a new condition to obtain the existence of zero points for set-valued operators. The condition is as follows:

$$-Tx \subset (N_C(x)\setminus\{0\})^c \quad \text{for all} \quad x \in C, \tag{1.2}$$

where $N_C(x)$ is the normal cone to C at x and D^c is the complement of the set D.

The purpose of this paper is to study the existence of zero points for pseudomonotone operators in Banach spaces. Using the conditioin (1.2) and the techniques used by Shih and Tan [23] and Yao [26], we first prove an existence theorem for pseudomonotone operators in a Banach space. We apply this result to obtain the existence of zero points of the mappings with a coercive condition which is related to Browder [5] and Minty [21]. We also show that the condition (1.2) is a necessary and sufficient condition for the existence of zero points of the mappings.

2 Preliminaries

Let X and Y be topological spaces and let T be a set-valued operator of X into Y. Then T is said to be upper semicontinuous at $x_0 \in X$ if for any neighborhood V of Tx_0, there exists a neighborhood U of x_0 such that $Ty \subset V$ for all $y \in U$. T is said to be upper semicontinuous if it is upper semicontinuous at each point $x \in X$.

Let E be a real Banach space and let E^* be the dual of E. We denote the value of $x^* \in E^*$ at $x \in E$ by $\langle x, x^*\rangle$. The duality mapping $J : E \to 2^{E^*}$ is defined by $J(x) = \{x^* \in E^*: \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}$ for all $x \in E$. Some properties of the duality mapping may be found in [6,24,25]. A Banach space E is said to be strictly convex if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then a Banach space E is said to be smooth if

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for all $x, y \in U$.

Let C be a nonempty closed convex subset of a Banach space E. Then the normal cone $N_C(x)$ of C at $x \in C$ is defined by

$$N_C(x) = \{x^* \in E^* : \langle x - y, x^*\rangle \geq 0 \quad \text{for all } y \in C\} \quad \text{for all } x \in C. \tag{2.1}$$

Let $T : C \to 2^{E^*}$ be a set-valued operator. T is said to be pseudomonotone (in the sense of Karamardian [16]) if for any $x, y \in C$, $x^* \in Tx$ and $y^* \in Ty$,

$$\langle x - y, y^*\rangle \geq 0 \ \text{ implies } \ \langle x - y, x^*\rangle \geq 0.$$

It is easy to see that, if T is monotone, i.e., for any $x, y \in C$, $x^* \in Tx$ and $y^* \in Ty$,

$$\langle x - y, x^* - y^*\rangle \geq 0,$$

then T is pseudomonotone. We denote the set $\{u \in C : 0 \in Tu\}$ by $T^{-1}0$. A point $u \in T^{-1}0$ is called a zero point of T. For $x, y \in C$, let $[x, y]$ denote the line segment $\{(1 - t)x + ty : t \in [0, 1]\}$. The operator T is said to be an upper hemicontinuous if for any $x, y \in C$, it is upper semicontinuous operator from $[x, y]$ to E^*, where E^* has the weak* topology.

Let E be a reflexive, strictly convex and smooth Banach space. For any $x \in E$, there exists a unique point $x_0 \in C$ such that

$$\|x_0 - x\| = \min_{y \in C} \|y - x\|.$$

The mapping $P_C : E \to C$ defined by $P_C x = x_0$ is called the metric projection from E onto C. For each $x \in E$, $P_C x$ satisfies

$$\langle P_C x - y, J(x - P_C x)\rangle \geq 0 \quad \text{for all} \quad y \in C \tag{2.2}$$

(see [1,2] for metric projections).

We first give some examples of mappings which satisfy the condition (1.2). As we shall see below, the condition (1.2) is closely related to the geometry of Banach spaces, the fixed point theory and the monotone operator theory.

Example 2.1 According to a result of authors [20, pp. 253–254], the duality mapping $J : E \to 2^{E^*}$ satisfies the condition (1.2) on the unit ball of E.

Example 2.2 Let H be a Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$ and C a nonempty closed convex subset of H. A mapping $S : C \to C$ is called nonexpansive on C if $\|Sx - Sy\| \leq \|x - y\|$ for all $x, y \in C$. If S has a fixed point, that is, there exists $u \in C$ such that $Su = u$, then

$$(S - I)x \in (N_C(x)\backslash\{0\})^c \tag{2.3}$$

for all $x \in C$. In fact, assume that there exists $x_0 \in C$ such that $(S - I)x_0 \in N_C(x_0)\backslash\{0\}$. Then $(S - I)x_0 \neq 0$ and

$$\langle y - x_0, (S - I)x_0\rangle \leq 0$$

for all $y \in C$. So, we have

$$\begin{aligned}
\|x_0 - u\|^2 &= \|x_0 - Sx_0 + Sx_0 - u\|^2 \\
&= \|x_0 - Sx_0\|^2 + 2\langle x_0 - Sx_0, Sx_0 - u\rangle + \|Sx_0 - u\|^2 \\
&= \|Sx_0 - u\|^2 + 2\langle x_0 - Sx_0, x_0 - u\rangle - \|x_0 - Sx_0\|^2 \\
&< \|Sx_0 - u\|^2 \\
&\leq \|x_0 - u\|^2.
\end{aligned}$$

This is a contradiction. Therefore, we obtain (2.3).

Example 2.3 Let E be a reflexive, strictly convex and smooth Banach space. An operator $A : E \to 2^{E^*}$ is called maximal monotone if A is monotone and $A = B$ whenever $B : E \to 2^{E^*}$ is monotone and $G(A) \subset G(B)$, where $G(A) = \{(x, x^*) \in E \times E^* : x^* \in Ax\}$. In our setting, we know from [24] that a monotone operator A is maximal if and only if $R(J + rA) = E^*$ for all $r > 0$. Further, for any $x \in E$, there exists a unique $x_r \in E$ such that

$$0 \in J(x_r - x) + rAx_r,$$

where J is the duality mapping. Define $J_r x = x_r$ and $A_r x = \frac{1}{r}J(x - x_r)$ for all $x \in E$. Then, $A_r x \in AJ_r x$; see [24]. Let $u \in A^{-1}0$ and $B[u] = \{z \in E : \|u - z\| \leq 1\}$. Then

$$-A_r x \in \left(N_{B[u]}(x) \setminus \{0\}\right)^c \tag{2.4}$$

for all $x \in B[u]$. In fact, assume that there exists $x_0 \in B[u]$ such that $-A_r x_0 \in N_{B[u](x_0)} \setminus \{0\}$. Then $-A_r x_0 \neq 0$ and

$$\langle x_0 - y, -A_r x_0 \rangle \geq 0$$

for all $y \in B[u]$. This implies $\langle x_0 - u, -A_r x_0 \rangle \geq 0$. Since $\langle x_0 - J_r x_0 + J_r x_0 - u, -A_r x_0 \rangle \geq 0$, we have

$$\begin{aligned}\langle J_r x_0 - u, -A_r x_0 \rangle &\geq \langle x_0 - J_r x_0, A_r x_0 \rangle \\ &= \left\langle x_0 - J_r x_0, \tfrac{1}{r} J(x_0 - J_r x_0) \right\rangle \\ &= \frac{1}{r} \|x_0 - J_r x_0\|^2,\end{aligned}$$

and hence

$$\langle J_r x_0 - u, -A_r x_0 \rangle \geq \frac{1}{r} \|x_0 - J_r x_0\|^2. \tag{2.5}$$

By the monotonicity of A, we have

$$0 \geq \langle J_r x_0 - u, -A_r x_0 \rangle.$$

This together with (2.5) that $\frac{1}{r}\|x_0 - J_r x_0\|^2 = 0$ and hence $x_0 = J_r x_0$. Since $-A_r x_0 \neq 0$, this is a contradiction. Therefore, we obtain (2.4).

3 Existence theorem

In this section, we obtain an existence theorem for pseudomonotone operators in a Banach space.

We need the following two theorems due to Fan [12] and Kneser [18], respectively.

Theorem A *Let Y be a convex subset of a Hausdorff topological vector space, and let $\emptyset \neq X \subset Y$. Let $F : X \to 2^Y$ be an operator such that $F(x)$ is closed for each $x \in X$ and the convex hull of every subset $\{x_1, x_2, \ldots, x_n\}$ of X is contained in $\cup_{i=1}^n F(x_i)$. If there exists $x_0 \in X$ such that $F(x_0)$ is compact, then $\cap_{x \in X} F(x) \neq \emptyset$.*

Theorem B *Let X be a nonempty convex subset of a vector space, let Y be a nonempty compact convex subset of a Hausdorff topological vector space and let f be a real valued function of $X \times Y$ satisfying the following conditions*:

(1) *For each $x \in X$, the function $y \mapsto f(x, y)$ is lower semicontinuous and convex*;
(2) *for each $y \in Y$, the function $x \mapsto f(x, y)$ is concave.*

Then

$$\min_{y \in Y} \sup_{x \in X} f(x, y) = \sup_{x \in X} \min_{y \in Y} f(x, y).$$

We also need the following lemma due to Shih and Tan [23].

Lemma A *Let X be a nonempty convex subset of a Banach space E, let $T : X \to 2^{E^*}$ be an operator such that each Tx is a nonempty weak* compact subset of E^* and let T be upper hemicontinous. Then for $\hat{y} \in X$, the inequality*

$$\sup_{x^* \in Tx} \left\langle \hat{y} - x, x^* \right\rangle \leq 0 \quad \textit{for all} \quad x \in X,$$

implies that

$$\inf_{y^*\in T\hat{y}} \langle \hat{y}-x, y^*\rangle \le 0 \quad \textit{for all} \quad x \in X.$$

Using Theorems A and B and Lemma A, we have the following result.

Theorem 3.1 *Let X be a nonempty closed convex subset of a Banach space E and let $T : X \to 2^{E^*}$ be a pseudomonotone operator such that each Tx is a nonempty weak* compact convex subset of E^* and T is upper hemicontinuous. If there exists a nonempty weakly compact convex subset C of X such that*

$$-Tx \subset (N_C(x)\backslash\{0\})^c \quad \textit{for all} \quad x \in C, \tag{3.1}$$

then $T^{-1}0 \neq \emptyset$.

Proof We define two set-valued operators F and G of C into itself by

$$Fx = \left\{ y \in C : \inf_{y^*\in Ty} \langle y-x, y^*\rangle \le 0 \right\}$$

and

$$Gx = \left\{ y \in C : \sup_{x^*\in Tx} \langle y-x, x^*\rangle \le 0 \right\}$$

for every $x \in C$. Then the convex hull of every finite subset of C is contained in $\cup_{i=1}^n Fx_i$. In fact, if this is not true, then there exist $\{x_1, x_2, \ldots, x_n\} \subset C$ and $\overline{x} \in \mathrm{co}\{x_1, x_2, \ldots, x_n\}$ such that $\overline{x} \notin \cup_{i=1}^n Fx_i$, where co$D$ is the convex hull of the set D. Since $\overline{x} \in \mathrm{co}\{x_1, x_2, \ldots, x_n\}$, $\overline{x} = \sum_{i=1}^n \lambda_i x_i$ for some $\lambda_i \ge 0$ $(i = 1, 2, \ldots, n)$ with $\sum_{i=1}^n \lambda_i = 1$. This implies that

$$\begin{aligned} 0 &= \inf_{y^*\in T\overline{x}} \langle \overline{x}-\overline{x}, y^*\rangle \\ &= \inf_{y^*\in T\overline{x}} \left\langle \overline{x} - \sum_{i=1}^n \lambda_i x_i, y^* \right\rangle \\ &= \inf_{y^*\in T\overline{x}} \left(\sum_{i=1}^n \lambda_i \langle \overline{x}-x_i, y^*\rangle \right) \\ &\ge \sum_{i=1}^n \lambda_i \inf_{y^*\in T\overline{x}} \langle \overline{x}-x_i, y^*\rangle \\ &> 0, \end{aligned}$$

which is a contradiction.

We next show that $\cap_{x\in C} Fx = \cap_{x\in C} Gx$. We first show that $\cap_{x\in C} Fx \subset \cap_{x\in C} Gx$. Let $x \in C$ and $y \in Fx$. Since Ty is weak* compact, there exists $z^* \in Ty$ such that

$$\langle y-x, z^*\rangle = \inf_{y^*\in Ty} \langle y-x, y^*\rangle \le 0.$$

Since T is pseudomonotone,

$$\langle y-x, x^*\rangle \le 0 \quad \text{for all} \quad x^* \in Tx,$$

that is,

$$\sup_{x^*\in Tx} \langle y-x, x^*\rangle \le 0.$$

It follows that $Fx \subset Gx$ for each $x \in C$, and hence $\cap_{x\in C} Fx \subset \cap_{x\in C} Gx$. On the other hand, let $y \in \cap_{x\in C} Gx$. Then we have $\sup_{x^*\in Tx} \langle y - x, x^*\rangle \le 0$ for each $x \in C$. By Lemma A, we have that $\inf_{y^*\in Ty} \langle y - x, y^*\rangle \le 0$ for each $x \in C$, and hence $y \in \cap_{x\in C} Fx$. It follows that $\cap_{x\in C} Gx \subset \cap_{x\in C} Fx$. Therefore, we obtain that $\cap_{x\in C} Gx = \cap_{x\in C} Fx$.

Next, we show that $\cap_{x\in C} Fx \neq \emptyset$. Since C is weakly compact, $\overline{Fx}^w$ is weakly compact for each $x \in C$, where $\overline{D}^w$ is the weak closure of the set D. By Theorem A, $\cap_{x\in C} \overline{Fx}^w \neq \emptyset$. On the other hand, for each $x \in C$, Gx is weakly closed, that is $\overline{Fx}^w \subset Gx$. It follows that

$$\emptyset \neq \cap_{x\in C} \overline{Fx}^w \subset \cap_{x\in C} Gx = \cap_{x\in C} Fx. \tag{3.2}$$

By (3.2), there exists $y_0 \in C$ such that

$$\sup_{x\in C} \inf_{y^*\in Ty_0} \langle y_0 - x, y^*\rangle \le 0.$$

Since Ty_0 is a weak* compact convex subset of E^*, by Theorem B, we have

$$\inf_{y^*\in Ty_0} \sup_{x\in C} \langle y_0 - x, y^*\rangle = \sup_{x\in C} \inf_{y^*\in Ty_0} \langle y_0 - x, y^*\rangle,$$

and hence

$$\inf_{y^*\in Ty_0} \sup_{x\in C} \langle y_0 - x, y^*\rangle \le 0.$$

Since Ty_0 is weak* compact, there exists $y_0{}^* \in Ty_0$ such that

$$\sup_{x\in C} \langle y_0 - x, y_0{}^*\rangle \le 0.$$

It follows that

$$\langle y_0 - x, -y_0{}^*\rangle \ge 0 \quad \text{for all} \quad x \in C.$$

This implies that $-y_0^* \in N_C(y_0)$. If $-y_0^* \neq 0$, then

$$-y_0^* \in N_C(y_0)\setminus\{0\}.$$

From (3.1) and $-y_0^* \in -Ty_0$, this is a contradiction. Consequently, we obtain that $0 = -y_0^* \in -Ty_0$, and hence $0 \in Ty_0$. □

Remark 3.1 The idea of the proof of Theorem 3.1 is known under the assumptions that T satisfies coercivity condition and E is reflexive (see for example [23,26]).

Using Theorem 3.1, we obtain an existence result in a reflexive Banach space.

Theorem 3.2 *Let E be a reflexive Banach space and let $T : E \to 2^{E^*}$ be a pseudomonotone operator such that each Tx is a nonempty weakly compact convex subset of E^* and T is upper hemicontinuous. If there exists $x_0 \in E$ such that*

$$\lim_{\|y\|\to\infty} \left(\inf_{y^*\in Ty} \frac{\langle y - x_0, y^*\rangle}{\|y\|} \right) = \infty, \tag{3.3}$$

then $T^{-1}0 \neq \emptyset$.

Proof From (3.3), there exists $r > 0$ such that

$$x_0 \in B_r \quad \text{and} \quad \inf_{y^* \in Ty} \frac{\langle y - x_0, y^* \rangle}{\|y\|} > 0 \quad \text{for all} \quad y \in E \quad \text{with} \quad \|y\| > r,$$

where $B_r = \{z \in E : \|z\| \leq r\}$. In particular, if $\delta > r$, then

$$x_0 \in B_\delta \quad \text{and} \quad \inf_{y^* \in Ty} \langle y - x_0, y^* \rangle > 0 \quad \text{for all } y \in \partial B_\delta,$$

where $B_\delta = \{z \in E : \|z\| \leq \delta\}$. It follows that

$$-Ty \subset (N_{B_\delta}(y) \backslash \{0\})^c \quad \text{for all} \quad y \in \partial B_\delta. \tag{3.4}$$

On the other hand, since $(N_{B_\delta}(y) \setminus \{0\})^c = E^*$, whenever $\|y\| < \delta$, it follows from (3.4) that

$$-Ty \subset (N_{B_\delta}(y) \backslash \{0\})^c \quad \text{for all } y \in B_\delta.$$

Since E is reflexive, B_δ is a weakly compact convex subset of E. It follows from Theorem 3.1 that $T^{-1}0 \neq \emptyset$. □

Remark 3.2 Condition (3.3) was studied by Browder [5] and Minty [21] under the assumption that T is a single-valued monotone operator.

4 Necessary and sufficient condition

In this section, we prove that the condition (1.2) is a necessary and sufficient condition for the existence of zero points of a pseudomonotone operator under the assumptions that $X = E$ and E is reflexive. In particular, Asplund [3] has shown that a reflexive Banach space E has an equivalent norm such that E is a strictly convex and smooth Banach space. Hence, we assume that the norm on E has these properties.

Proposition 4.1 *Let E be a reflexive, strictly convex and smooth Banach space and let $T : E \to 2^{E^*}$ be a pseudomonotone operator such that $T^{-1}0 \neq \emptyset$ and let $u_0 \in T^{-1}0$ and $r > 0$. Then*

$$-Tx \subset (N_{B_r[u_0]}(x) \setminus \{0\})^c \quad \textit{for all} \quad x \in B_r[u_0],$$

where $B_r[u_0] = \{x \in E : \|u_0 - x\| \leq r\}$.

Proof Let $u_0 \in T^{-1}0$ and $r > 0$. Assume that there exists $z \in B_r[u_0]$ such that

$$-Tz \cap (N_{B_r[u_0]}(z) \backslash \{0\}) \neq \emptyset,$$

that is, there exists $z^* \in Tz$ such that

$$-z^* \in N_{B_r[u_0]}(z) \backslash \{0\} \tag{4.1}$$

Since $N_{B_r[u_0]}(x) \setminus \{0\} = \emptyset$ whenever $\|x\| < r$, we may assume $z \in \partial B_r[u_0]$ without loss of generality.

From (4.1),

$$-z^* \neq 0 \quad \text{and} \quad \langle z - y, -z^* \rangle \geq 0 \quad \text{for all} \quad y \in B_r[u_0],$$

and hence

$$-z^* \neq 0 \quad \text{and} \quad \left\langle z - y, J(z - J^{-1}z^* - z) \right\rangle \geq 0 \quad \text{for all} \quad y \in B_r[u_0].$$

It follows from (2.2) that

$$z = P_{B_r[u_0]}(z - J^{-1}z^*).$$

In particular, $z - J^{-1}z^* \notin B_r[u_0]$. In fact, if $z - J^{-1}z^* \in B_r[u_0]$, then

$$\begin{aligned} z &= P_{B_r[u_0]}(z - J^{-1}z^*) \\ &= z - J^{-1}z^*, \end{aligned}$$

and hence $-z^* = 0$, which is a contradiction.

Let $t = \frac{r}{\|z - J^{-1}z^* - u_0\|}$ and $w_0 = t(z - J^{-1}z^* - u_0) + u_0 \in \partial B_r[u_0]$. We next show that

$$w_0 = P_{B_r[u_0]}(z - J^{-1}z^*).$$

For each $y \in B_r[u_0]$, we have

$$\begin{aligned} \left\langle w_0 - y, J(z - J^{-1}z^* - w_0)\right\rangle &= \left\langle w_0 - y, J(z - J^{-1}z^* - u_0 + u_0 - w_0)\right\rangle \\ &= \left\langle w_0 - y, J(\tfrac{1}{t}(w_0 - u_0) + u_0 - w_0)\right\rangle \\ &= \frac{1-t}{t}\langle w_0 - u_0 + u_0 - y, J(w_0 - u_0)\rangle \\ &= \frac{1-t}{t}(\|w_0 - u_0\|^2 + \langle u_0 - y, J(w_0 - u_0)\rangle) \\ &\geq \frac{1-t}{t}(r^2 - \|y - u_0\|\|w_0 - u_0\|) \\ &= \frac{1-t}{t}r(r - \|y - u_0\|) \geq 0. \end{aligned}$$

It follows from (2.2) that $w_0 = P_{B_r[u_0]}(z - J^{-1}z^*)$. This implies that

$$t(z - J^{-1}z^* - u_0) + u_0 = w_0 = P_{B_r[u_0]}(z - J^{-1}z^*) = z,$$

and hence

$$z^* = \frac{1-t}{t}J(u_0 - z).$$

Since T is pseudomonotone, from $z^* \in Tz$ and $u_0 \in T^{-1}0$, we have

$$0 \leq \langle z - u_0, z^*\rangle = \left\langle z - u_0, \tfrac{1-t}{t}J(u_0 - z)\right\rangle = -\frac{1-t}{t}\|z - u_0\|^2 < 0,$$

which is a contradiction. Therefore, we obtain that

$$-Tx \subset (N_{B_r[u_0]}(x) \setminus \{0\})^c \quad \text{for all} \quad x \in B_r[u_0].$$

□

By Theorem 3.1 and Proposition 4.1, we have the following necessary and sufficient condition.

Theorem 4.1 *Let E be a reflexive, strictly convex and smooth Banach space and let $T : E \to 2^{E^*}$ be a pseudomonotone operator such that each Tx is a nonempty weakly compact convex subset of E^* and T is upper hemicontinuous. Then the following are equivalent*:

(1) $T^{-1}0 \neq \emptyset$;
(2) *there exists a nonempty weakly compact convex subset* C *of* E *such that*

$$-Tx \subset (N_C(x)\setminus\{0\})^c \quad \textit{for all} \quad x \in C.$$

Proof It follows from Theorem 3.1 that (2) implies (1). Let $u \in T^{-1}0$ and $r > 0$. By Proposition 4.1, we have that

$$-Tx \subset (N_{B_r[u]}(x)\setminus\{0\})^c \quad \text{for all } x \in B_r[u].$$

Since E is reflexive, $B_r[u]$ is a weakly compact convex subset of E. Hence we obtain that (1) implies (2). □

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